The use of Chebyshev Polynomials to boundedeness of solutions of difference equations

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Abstract

The boundedness and global attractivity of the nonnegative solutions of a nonlinear difference equation is investigated by using the (extended) Chebyshev polynomials. The paper is motivated by an open problem proposed in [V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order and Applications,* Kluwer Academic Publishers, Dordrecht, 1993].

1 Introduction

By using the (generalized) Chebyshev polynomials in this article we obtain some results concerning boundedness of the solutions of a difference equation.

Recently there has been a lot of activity concerning the asymptotic behavior of solutions of difference equations. Difference equations appear as natural descriptions of observed evolution phenomena as well as in the study of discretization methods for differential equations. And although difference equations seem to be elementary in their own presentation, their theory is a lot richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation by the usual Euler's scheme, may have a phenomenon often called the appearance of *ghost* solutions, or the existence of *chaotic orbits* that can only happen for higher order differential equations. Chaos and fractals are at the center of attention nowadays and difference equations theory is what gives birth to both of them. Take, for instance, the simple difference equation

$$x_{n+1} = T(x_n) := 1 - 2\left|x_n - \frac{1}{2}\right|,$$

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with initial values in the interval [0, 1] of the real line. It is easy to show that it has periodic solutions of any period. (For each positive integer k the function $T^{(k)}$ admits fixed points). Generally speaking, this is a phenomenon shared by all difference equations that possess a 3-cycle and it was discovered in [1]. On the other hand it is well known, even from the end of the 19th century, that linear difference equations and linear differential equations with polynomial coefficients are related through a formal Mellin transformation. Indeed, let

$$\sum_{h=0}^{M}\sum_{l=0}^{m} a_{hl} s^h \left(s\frac{d}{ds}\right)^l y = 0, \tag{D}$$

be a linear differential equation and let τ denote the shift operator

$$\tau(x)(s) := x(s+1).$$

The formal Mellin transformation is defined by the substitutions

$$s \to \tau, \; s\frac{d}{ds} \to -t$$

to the differential equation D. Then the linear difference equation

$$\sum_{h=0}^{M}\sum_{l=0}^{m} (-1)^l a_{hl}(t+h)^l x(t+h) = 0, \tag{Δ}$$

is generated. If

$$\sum_{n=-\infty}^{+\infty} y_n s^{-n}$$

is a formal solution of (D), then the coefficients y_n satisfy the difference equation (Δ) and visa-versa, namely analytic Mellin transforms of solutions of (D) are used to represent solutions of (Δ). For this situation one can consult [2].

Therefore the theory of difference equations is interesting in itself and assumes great importance in the present world.

One of the properties which we investigate in difference equations, is boundedeness of the solutions. In terms of dynamical systems this is equivalent to compactness of the solutions with respect to the discrete shifting semi-flow. More facts on this topic can be found in [3]. See, also, [4], [5], [6], [7]. More applications of these items are given in the recent book [8].

To show that one of the main problems in investigating such equations is to establish boundedness of the solution we shall deal with a difference equation of the form

$$x_{n+1} = x_n^{\gamma} f(x_{n-k}), \tag{1}$$

where $\gamma > 0$ and k is any positive integer. The problem is motivated by an open problem stated in the monograph [9] and the work [10] The results are presented in [11] and extend those given elsewhere and in [12]. For the same function $f(x) = b/(1+x^2)$ and exponent $\gamma = 2$ the problem was discussed in [13], where the condition

$$(k+1)^{k+1} > (2k)^k$$

was used to guarantee boundedness of the solutions. But it is easy to see that this condition is true only for $k = 1, 2, 3$. Hence it is not so general as it seems to be.

2 The main results

To proceed consider the following linear recursive relation

$$p_{n+1}(a,x) = ap_n(a,x) - xp_{n-k}(a,x), \quad n = 1, 2, ..., \tag{2}$$

where $a > 0$ and $x \in \mathbf{R}$. Associating Equ. (2) with the initial values

$$p_j(a,x) = a^j, \quad j = 0, 1, ..., k,$$

it is obvious that the solution $p_n(a,x)$ is a polynomial. For $k = 1$ the solution of the difference equation (2) is given by

$$p_n(a,x) = 2^n x^{\frac{n}{2}} U_n(a2^{-1}x^{\frac{-1}{2}}),$$

where $U_n(x)$ is the nth Chebychev polynomial of the second kind

$$U_n(x) = \frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(n-k)!}{k!(n-2k)!}(2x)^{n-2k}.$$

On the other hand it is very well known that the polynomial $U_n(x)$ can be given via the solution p_n by the formula

$$U_n(x) = p_n(x, 2^{-2}).$$

The basic things we use below are the polynomials of the form

$$q_n(x) := p_n(\gamma, x),$$

where γ is the exponent in Equ. (1). For these items we have the following information:

Lemma. *Assume that s_n is the least positive solution of the polynomial $q_n(x)$. Then we have the following facts:*

(i) The sequence (s_n) is strictly decreasing, and

(ii) For each integer $n > k$ and all $j = 0, 1, \cdots, n$ we have

$$q_{n+1}(s_n) < 0$$

$$q_j(s_{n+1}) > 0.$$

Remark. From the previous results we conclude that the sequence (s_n) of the least positive zeroes of the polynomials $q_n(x)$ converges to a point $l \geq 0$. It would be very useful to have information about it. Later on we shall see that in case $k = 1$ this limit is equal to $\frac{\gamma^2}{4,}$ but for the general case the question is open.

Theorem 1. *Assume that $f : \mathbf{R}^+ \to \mathbf{R}^+$ is a bounded function. If there exist $n > k$ and $r > 0$ as well as a root $s_n > 0$ of the polynomial $q_n(x)$ such that*

$$r \geq s_n, \tag{3}$$

and

$$\sup_{x>0} x^r f(x) < +\infty \tag{4}$$

then any (positive) solution of equation (1) is bounded.

Sketch of the Proof: Let $M > 0$ be such that $f(x) \leq M$ and $x^r f(x) \leq M$, for all $x \geq 0$. Then it is easy to see that for each $s \in [0, r]$ it holds

$$x^s f(x) \leq M.$$

Consider an index $n > k$ satisfying inequality (3). For simplicity set $\lambda := s_n$ and then, because of Lemma 1, we have

$$q_n(\lambda) = 0$$

and

$$q_j(\lambda) > 0,$$

for all $j = 0, 1, 2, ..., n - 1$.

Consider a solution (x_j) of Equ. (1). Take any $m \geq n + 1$ and for simplicity set

$$z_j := f(x_j), \ \ y_j := x_j^\lambda f(x_j) \ \text{ and } \ t_j := q_j(\lambda).$$

By making some manipulations on Equ. (1) we get

$$
\begin{aligned}
x_{m+1} &= \Pi_{j=0}^{n-k-1}(y_{m-(k+j)})^{t_j}(x_{m-(n-1)})^{t_n}\Pi_{j=n-k}^{n-1}(z_{m-(k+j)})^{t_j} \\
&\leq M^{\sum_{j=0}^{n-k-1} t_j}\Pi_{j=n-k}^{n-1}(z_{m-(k+j)})^{t_j} \\
&\leq M^{\sum_{j=0}^{n-1} t_j}.
\end{aligned}
$$

This proves the theorem.

Corollary. *Assume that $f : \mathbf{R}^+ \to \mathbf{R}^+$ is a bounded function. If there exists*

$$r > \frac{\gamma^2}{4} \tag{5}$$

satisfying relation (4), *then any (positive) solution of the difference equation*

$$x_{n+1} = x_n^{\gamma} f(x_{n-1})$$

is bounded. Condition (5) *is sharp, in the sense that it can not be weakened.*

Proof. It is well known that the Chebyshev polynomials of the second kind U_n are given by the type

$$U_n(x) = \frac{1}{2^n}\frac{sin[(n+1)arc\cos x)]}{\sqrt{1-x^2}}.$$

Therefore the zeroes of the polynomial U_n are the numbers

$$cos(\frac{j\pi}{n+1}),\ \ j = 1, 2, ..., n.$$

The greatest of them is the number $cos(\frac{\pi}{n+1})$. Hence we get

$$s_n = \frac{\gamma^2}{4cos(\frac{\pi}{n+1})},$$

which is a decreasing sequence (however, this illustrates the result of Lemma 1) and it tends to $\frac{\gamma^2}{4}$. Now the first part of the proof follows by choosing s_n with

$$r \geq s_n > \frac{\gamma^2}{4}$$

and applying Theorem 1.

The sharpness of condition (5) can be proved by the use of the difference equation

$$x_{n+1} = \frac{2x_n^2}{1+x_{n-1}}.$$

Here we have

$$f(x) := \frac{2}{1+x} \text{ and } \gamma = 2.$$

Thus the supremum of all $r > 0$ which make $x^r f(x)$ bounded is $r_0 = 1$, which is not (strictly) greater than $\frac{\gamma^2}{4} = 1$. It can be shown that there exist unbounded solutions of this equation.

Theorem 2. *Assume that the function $f : [0, +\infty) \to [0, +\infty)$ is continuous and decreasing and such that*

$$f(0) < 1.$$

If there exist a real $u > 0$ and an index $n > k + 1$ such that

$$p_n(\gamma, u) \leq 1,$$

and

$$0 \leq p_j(\gamma, u), \ \ j = 0, 1, \cdots, n-1$$

and

$$\sup_{x>0} x^u f(x) \leq 1,$$

then the origin attracts all positive solutions of the difference equation

$$x_{n+1} = x_n^\gamma f(x_{n-k}). \tag{1}$$

Sketch of the Proof: For each $j = 0, 1, 2, ...$ set

$$\zeta_j := p_j(\gamma, u), \ \ z_j := f(x_j) \text{ and } y_j := x_j^u f(x_j).$$

Then for all $m > n > k$ from Equ. (1) we can obtain

$$\begin{aligned} x_{m+1} &= \Pi_{j=0}^{n-k-1} (y_{m-(k+j)})^{\zeta_j} (x_{m-(n-1)})^{\zeta_n} \Pi_{j=n-k}^{n-1} (z_{m-(k+j)})^{\zeta_j} \\ &\leq (x_{m-(n-1)})^{\zeta_n} \Pi_{j=n-k}^{n-1} (z_{m-(k+j)})^{\zeta_j} \\ &\leq (x_{m-(n-1)})^{\zeta_n} f(0)^{\sum_{j=n-k}^{n-1} \zeta_j} < (x_{m-(n-1)})^{\zeta_n} \end{aligned}$$

and therefore

$$x_m < x_{m-n}, \tag{6}$$

for all $m > n + 1$. From (6) we conclude that the limit

$$l_m := \lim_{j \to +\infty} x_{m+jn}$$

exists. (Actually the sequence (l_m) can be extended to a full limiting sequence. More facts on this item as well as its use can be found in [3-7].) By the continuity of the function f the sequence (l_m) satisfies Equ. (1). If a term l_m is zero, then $l_j = 0$, for all $j \geq m$. So, assume that it holds $l_m > 0$, for all m. Moreover it follows that the limit

$$l_m^1 := \lim_{j \to +\infty} l_{m+jn}$$

exists. Now we observe that

$$l_m < x_{m+in},$$

for all $i = 0, 1, ...$ and so

$$l_{m+jn} < x_{m+jn+in},$$

for all $i = 0, 1, \cdots$ and $j = 0, 1, 2...$ Taking the limits first with respect to j and then with respect to i we obtain

$$l_m^1 \leq l_m.$$

Again we have

$$l_m < x_{m+jn+in}$$

for all $i = 0, 1, 2...$ and $j = 0, 1, 2...$ Thus

$$l_m \leq \lim_{i \to +\infty} x_{m+jn+in} = l_{m+jn},$$

which gives

$$l_m \leq l_m^1.$$

The two previous inequalities imply that the sequence (l_m) is constant, thus the limit

$$\lim_{j \to +\infty} x_j =: l$$

exists. This number is a (stationary) solution of Equ. (1). By using comparison facts as the above, we can easily show that $l = 0$ and the proof is complete.

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